UNIVERSITY OF CAMBRIDGE
Department of Computer Science and Technology

Avengers College

Computer Science Admissions Test

EXAMPLE QUESTIONS. *These are meant as practice material and have various degrees of difficulty. Some are inspired from past CSAT papers.*

Instructions:

- The test duration is 120 minutes. Section A has 8 questions. Section B has 12 questions which are more challenging and worth more marks.
- All questions attempted are marked. Your best 5 questions from each section are considered. Partial answers are taken into account. You can choose the questions to answer and their order.
- Write only on the work paper provided. Clearly label the question you are solving at the top of each page and start a new question on a new page (do not answer multiple questions on the same page). Answers without working may not gain full marks. You should show sufficient working to make your solutions clear to the Examiner, but these need not be extremely thorough.
- All paper must be handed in.
- Calculators, phones, watches, smart-glasses, other electronic devices or own paper are **not** permitted.
- **Do not** discuss any test questions with others (e.g. candidates at the same or another school, the Internet, or elsewhere), especially before March. You would disadvantage yourself.

It is recommended that you:

- take 5 minutes first to read through all questions,
- start with Section A and spend no more than 30 minutes on it,
- aim for 5 questions in each section; if you finish early then attempt more from Section B.

Good luck!

Section A — aim for 5 questions out of 8, of your choosing

1. How many 10 digit natural numbers do not contain the digit 6? Numbers cannot start with zero.

2. Alice and Bob want to know Carol's birthday. Carol gives them a list of 10 possible dates: March 11, 12, 15; April 13, 14; May 10, 12 and June 10, 11, 13. Carol then tells Alice the month of her birthday, and Bob the day of her birthday. Alice and Bob say the following:

   Alice: *I don't know Carol's birthday, but Bob doesn't know either.*
   Bob: *Previously I didn't know Carol's birthday, but now I do.*
   Alice: *I now also know Carol's birthday.*

   When is Carol's birthday? Explain your reasoning.

3. Let $a > 1$ be an integer. Give a non-integral expression in terms of a for $F(a) = \displaystyle\int_1^a (-1)^{\lfloor x \rfloor} \lfloor x \rfloor^{-1} dx$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

4. The parabola $y = x^2 + \frac{1}{4}$ is eventually intersected at two points by the line $y = tx$, where t is the time increasing linearly from 0 (hence the line rotates from a horizontal towards a vertical position). Determine the speed at which the segment linking the two intersection points grows.

5. A thin square piece of paper of side 1 is folded recursively. The first fold is along a diagonal of length $\sqrt{2}$, thus obtaining an isosceles triangle. All successive folds are along a segment of length l_i, with $i = 1, 2, \ldots$, that results in a new triangle that is similar (geometrically) to the previous one. Find the sum of all l_i as this process continues indefinitely.

6. Let $F_k = F_{k-1} + F_{k-2}$ where $F_0 = 0, F_1 = 1$. Define the series $s(x) = \sum_{k=1}^{\infty} F_k / x^k$ for $x > 2$. Show that $s(x) = \frac{x}{x^2 - x - 1}$.

7. The function f is defined recursively for all integers $n > 1, p > 0$ as follows:

$$\begin{aligned} f(1, p) &= C + p, \\ f(n, 1) &= C, \\ f(n, p) &= f(n-1, f(n, p-1)), \end{aligned}$$

   where $C > 0$ is a real constant. Give a non-recursive expression for $f(4, p)$. Proof is not required. *Hint:* You may want to try $f(2, p)$ first.

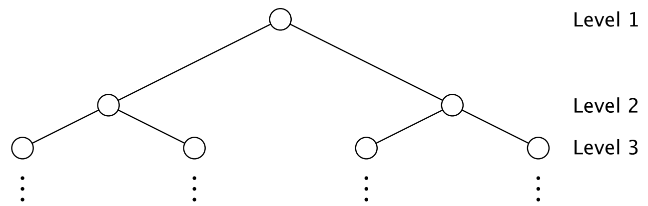
8. Find all functions $f : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$, such that for all $n = 1, 2, 3, \ldots$ the sum $f(1) + f(2) + \cdots + f(n)$ is equal to a perfect cube that is less than or equal to n^3.

Section B — aim for 5 questions out of 12, of your choosing

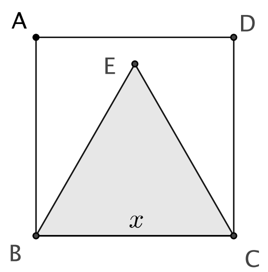
9. A robot on a Cartesian grid can perform three moves: F moves forward, R turns right 90 degrees, L turns left 90 degrees. The robot can be programmed and all its programs are recursive, apart from a few base cases. For example, a program could be $P_{n+1} = RP_nLQ_0R$ with $P_0 = F, Q_0 = F$. Design a recursive program P_n which makes the robot trace an equidistant square spiral as it moves on the grid. By "equidistant" we mean that the distance between any two closest parallel arms of the spiral is constant (but not necessarily equal to 1).

10. There are two tins each containing n biscuits. A tin is chosen at random and a biscuit is removed. This is repeated until one tin becomes empty. What is the probability that there are k biscuits left in one tin when the other becomes empty? Explain the steps of your solution.

11. You have a binary tree with n levels similar to the figure shown. On each level you draw a line passing through all nodes on that level. What is the total number of triangles formed in this way? Give a recursive expression for this number.



12. Consider the square $ABCD$ of side x, and the equilateral triangle BCE as in the figure shown. The square rotates clockwise around B until A overlaps E, then rotates around E until D overlaps C, and so on, until A retakes its initial position. Sketch the path traced by A and find its length. Give the length of the longest horizontal segment with end points on this path.



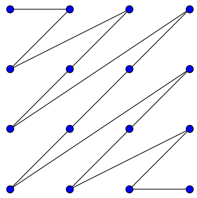
13. Sketch $(1 + x)^y = e$ for all real values. Take care to point out all key points and key behaviour.

14. There is a queue for admissions to The Avengers team. You are offered free admission if you are the first candidate in the queue who shares a birthday with a candidate earlier in the queue. You have the power to sneak into any position in the queue undetected. Which position in the queue gives you the highest probability of getting the free admission?

15. In base 10, or decimal, we use the digits from 0 to 9 to represent any positive integer. In base 2, or binary, we use the digits from 0 to 1 to do the same. An integer is called a *repunit* (repeated unit) if it can be written in some base using only the digit 1. Find all $n, p > 0$ such that a binary repunit with n digits is equal (in value) to a decimal repunit with p digits. Prove your answer.

16. Let S_i be a sequence of integers obeying $S_{(S_i+5)} = i - 1$, for $i = 0, 1, 2, \ldots$ and $S_0 = 0$. Give a non-recursive expression for *(a)* one such sequence S_i, *(b)* all possible such sequences S_i. Different expressions for different subsets of $i = 0, 1, \ldots$ are allowed, e.g. "i *is odd*" and "i *is even*".

17. The figure shows a non-overlapping trace on a 4×4 grid which visits all points exactly once. Imagine the same type of trace on an $n \times n$ grid, where n can be arbitrarily large. Using the fact that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity, show that the sum of all acute angles of the trace also diverges to infinity when n tends to infinity, despite the angles tending to 0 the closer the path gets to the grid's diagonal.



18. A poll with 2 choices is run among $n > 2$ participants. Each participant chooses at random. The poll shows the results for the 2 options as percentages rounded to the nearest integer, i.e. x rounded is $\lfloor x + 0.5 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For example, 1/3 would be shown as 33%. What is the probability that the sum of the 2 shown percentages does not add to 100?

19. There are n chemicals. One is magic, turning lead into gold in one day. You may *(i)* mix multiple chemicals with lead in one flask; *(ii)* add one chemical to multiple flasks. You have p flasks, and an unlimited supply of lead and of each of the n chemicals. What is the smallest number of days you need to guarantee you find the magic chemical?

20. The *Csatian* language has the following two rules: *(i) B is a word*, and *(ii) if x is a word then $BBBxxx$ is a word.*
(a) Give an expression for the number of Bs in a valid word.
(b) Suggest a third rule *(iii)* such that all multiples of 3 Bs are valid words, and the only other valid word is a single B.
(c) Replace rule *(iii)* with the following: *(iii) if $BBBBx$ is a word then x is a word.* Find all pairs (p, q) such that, for *all* integers $n \geq 0$, the expression $pn + q$ gives the lengths of all valid words.