Avengers College

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

Computer Science Admissions Test

EXAMPLE QUESTIONS. *These are meant as practice material and have various degrees of difficulty. Some are inspired from past CSAT papers.*

Write your full name in the box below as LAST NAME, FIRST NAME(S) in block capitals:

BANNER, BRUCE

Instructions:

- The test duration is 100 minutes. Section A has 8 questions. Section B has 12 questions which are more challenging and worth more marks.
- All questions attempted are marked. Your best 5 questions from each section are considered. Partial answers are taken into account. You can choose the questions to answer and their order.
- Write only on the blank paper provided and clearly label the question you are solving at the top of each page. Answers without working may not gain full marks. You should show sufficient working to make your solutions clear to the Examiner, but these need not be extremely thorough.
- All paper must be handed in.
- Calculators, phones, watches, smart-glasses, other electronic devices or own paper are **not** permitted.
- **Do not** discuss any test questions with others (e.g. candidates at the same or another College, the Internet, or elsewhere), especially before March. You would disadvantage yourself.

It is recommended that you:

- take 5 minutes first to read through all questions,
- start with Section A and spend no more than 30 minutes on it,
- aim for 5 questions in each section; if you finish early then attempt more from Section B.

Good luck!

Section A — aim for 5 questions out of 8, of your choosing

1. Produce a sketch of $|x|^n + |y|^n = 1$ for each $n \in \{1, 2, 1000\}$.

2. Show that 12 divides $n^4 - n^2$ for all positive integers n.

3. A 3 digit lock gives feedback when trying out a combination. What is the correct unlocking combination if the lock responds as follows for the following attempts:
   - `206` : *two numbers are correct but wrongly placed*
   - `738` : *no numbers are correct*
   - `682` : *one number is correct and correctly placed*
   - `614` : *one number is correct but wrongly placed*
   - `780` : *one number is correct but wrongly placed*

   Briefly justify your answer.

4. A hiker starts on a path at time $t = 0$ and reaches destination after 1 hour. During the hike, his velocity in km/h varies according to the function $v(t) = \cos(t\pi/2)$. Find the time in hours at which the hiker reaches the halfway distance between start and destination. *Hint:* $\sin(\pi/6) = 1/2$.

5. How many functions g can be defined from set $A = \{0, 1, \ldots, 2^n - 1, 2^n\}$ to set $B = \{0, \ldots, n\}$ such that $g(2^x) = x$ for all x in B?

6. A positive integer n is said to be triangular if $n = \sum_{i=0}^{k} i$ for some positive integer k. Given $8n + 1$ is a square number, show that n is triangular.

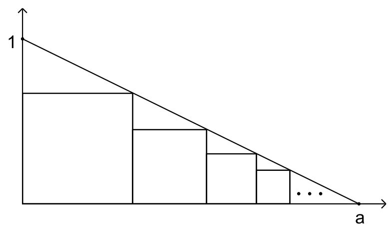
7. Let a_n, b_n be sequences of positive real numbers which satisfy $a_0 = b_0$ and the recursive matrix relation $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$. What is the simplest non-recursive form of the ratio b_n/a_n?

8. A whiteboard has p "A" symbols and m "B" symbols written on it. You choose any two symbols to erase and replace them with another one according to the following rules:
   - $AA \Rightarrow B$
   - AB or $BA \Rightarrow A$
   - $BB \Rightarrow B$

   If you continue to apply replacements for as long as possible, which values of p and m result in a single A remaining at the end?

Section B — aim for 5 questions out of 12, of your choosing

9. A line crosses the x and y axes at $(a, 0)$ and $(0, 1)$ respectively, where $a > 0$. Squares are placed successively inside the right angled triangle thus formed as in the figure below. What is the area enclosed by all squares when their number goes to infinity?



10. N circles in a plane all intersect each other such that every circle intersects every other circle at exactly 2 points. Find in terms of N the minimum and maximum number of disjoint closed regions that can be formed. *Hint:* You may wish to check your answers (visually) for greater N, e.g. $N = 4$.

11. Let $I_n = \displaystyle\int_{-\pi/2}^{\pi/2} \cos^n x \, dx$ for any non-negative integer n.
   (i) Find a recursive expression for I_n.
   (ii) Find the simplest non-recursive expression for I_{2n} that contains $\binom{2n}{n}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

12. A bank has a loan scheme with an annual interest rate of $0 < r < 1$ ($100r$ is the annual percentage rate). The bank charges interest every month. You take a loan of L pounds and wish to pay it back in exactly m months by making monthly payments of p pounds each. Find p.

13. n passengers board a plane with n seats. Each passenger has an assigned seat. The first passenger forgets and takes a seat at random. Every subsequent passenger sits in their assigned seat unless it is already taken, at which point they take a random seat. What is the probability (in terms of n) that:
   (i) Every passenger sits in the correct seat?
   (ii) At least one passenger sits in the correct seat?
   (iii) Exactly one passenger sits in the correct seat?

14. The *Seesat* language has the single-letter word A. Longer words are built by applying a sequence of the following rules: Rule r_1 says that if x is a word then Ax is a word; Rule r_2 says that if x and y are words then Bxy is a word. For example, the sequence $r_1(r_2(A, A))$ yields the word $ABAA$.
   (i) Give a valid sequence of rules that yields the word $AAABBAAA$.
   (ii) Write $N_A(x)$ and $N_B(x)$ for the number of A and B letters in the word x respectively. By expressing $N_A(x)$ and $N_B(x)$ in connection with the rules, prove mathematically that $N_A(x) > N_B(x)$ for any word x in the language.

15. Find all values of $x \geq 0$ in terms of k that satisfy $\lfloor kx \rfloor = (k+1)\lfloor x \rfloor$, where $k > 0$ is an integer and $\lfloor r \rfloor$ is the greatest integer less than or equal to r. You may wish to consider $k = 2$ first.

16. Give an example (i.e. equation) of a non-constant polynomial function of the smallest degree $f(x)$, such that all of the following hold: *(a)* it has at least one inflexion point, *(b)* all inflexion points have y-coordinate equal to 0, *(c)* all its roots are real, and *(d)* it is symmetric with respect to the y-axis.

17. You draw n cards at random from a standard deck of 52 playing cards. If the n cards contain either a J, Q or K then you win. If not then you put all n cards back in the deck, reshuffle and draw $n - 1$ new cards. You repeat until you win or until $n = 0$, when you lose. What are the probabilities of winning and losing if you start with *(a)* $n = 1$, *(b)* $n = 2$, *(c)* $n = 8$, or *(d)* $n = 41$?

18. Let A, B and C represent the number of tokens in three piles. A *game* between Alice and Bob starts with at most n tokens in each pile (i.e. $0 < A, B, C \leq n$) and consists of them taking turns. A *turn* consists of a player removing x tokens from one pile and y tokens from a different pile, with the constraint that $x + y > 0$. The player to remove the last set of tokens wins. Alice goes first in every game, and they play through all possible starting configurations of A, B, C. If Alice and Bob play in such a way that ensures their own number of wins is maximized, how many games does Alice win?

19. Let $A = \{0, 1, \ldots, 2^n - 1, 2^n\}$ and $B = \{0, \ldots, n\}$. How many functions g can be defined from A to B such that both of the following conditions hold:
    - for all $x \in B$ we have $g(2^x) = x$,
    - for all $y, z \in A$ with $y \leq z$ we have $g(y) \leq g(z)$.

20. The numbers $1, 2, ..., n$ are permuted (or shuffled). How many different permutations exist such that no two of the numbers $1, 2, 3$ are adjacent when $n = 5$ and $n = 6$? How about for arbitrary $n > 4$?